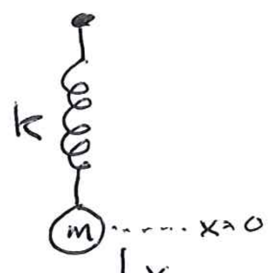
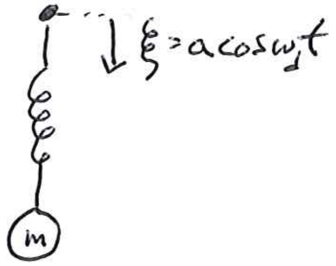
Forced harmonic oscillator (review + damping)

- Consider case of a mass hanging from a spring w/ no damping:



- This system exhibits simple harmonic motion described by: $m\frac{d^2x}{dt^2} = -kx$
- It oscillates at its natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$ w/ the form $x = A\cos\omega_0 t$

- Now consider what happens if we apply a force to periodically move the base of the spring up & down by an amount $\xi = a\cos\omega_d t$; ω_d = driving frequency

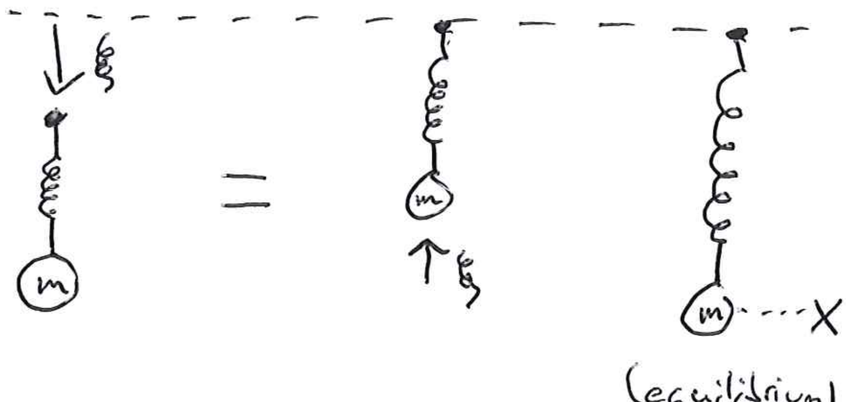


- This applied motion to the base of the spring dynamically extends & compresses the spring, which changes the force experienced by the mass

- This modifies the eq. of motion to give:

$$m\frac{d^2x}{dt^2} = -k(x-\xi)$$ (we take the down direction as positive)

- How does this make sense? Consider the reference frame of the mass. Moving the base of the spring down by an amount ξ provides an equivalent compression of the spring as moving the mass up by ξ. Therefore, replacing x with $x-\xi$ accurately captures this force.



- Can rearrange $m\frac{d^2x}{dt^2} = -k(x-\xi)$

$$\rightarrow \boxed{m\frac{d^2x}{dt^2} + kx = k\xi = ka\cos\omega_d t}$$

has units of force, so we call this the amplitude of the applied force $F_0 = ka$

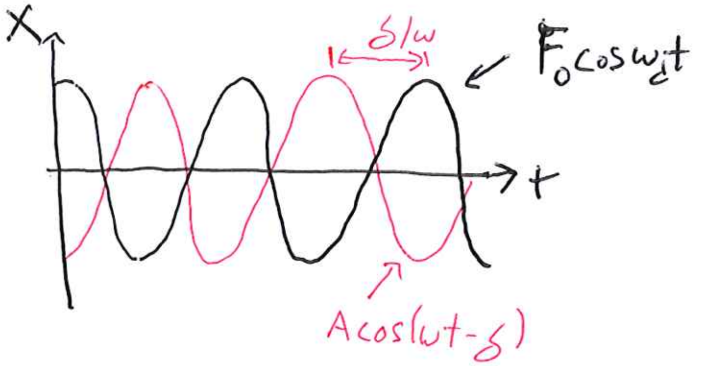
$$\rightarrow \boxed{m\frac{d^2x}{dt^2} + kx = F_0\cos\omega_d t}$$

since $\omega_0^2 = \frac{k}{m}$

$\rightarrow F_0 = m\omega_0^2 a$

- What is the solution to this eq. of motion?
  → Need to know this to understand how the mass will oscillate!

- Ansatz: $x(t) = A\cos(\omega t - \delta)$



↑ phase difference b/t driving force $F_0 \cos\omega_d t$ and mass displacement

minus: we will see b/c displacement lags behind drive

- Important! There are <u>3</u> frequencies to consider:

  ω_0: Natural Frequency of spring/mass system

  $$\omega_0 = \sqrt{\frac{k}{m}}$$

  - driving will not change this natural frequency!

  ω_d: Frequency of the drive

  ω: Frequency that mass oscillates at in driven system

* $\boxed{\omega = \omega_d}$ Systems <u>always</u> oscillate at the Frequency of the drive (just not always very well)

- One can solve eq. $m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega_d t$

w/ $x = A\cos(\omega t - \delta)$ (assuming ω not necessarily $= \omega_d$)

to prove $\omega = \omega_d$, but we will just accept that this is in fact the case.

- So for the rest of this lecture, we replace ω_d w/ ω, since we know they are the same. (The textbook does this w/o much explanation)

- Another important point: We will see that the amplitude of oscillation A depends on the driving frequency $\omega_d = \omega$, so our ansatz solution becomes:

$$\boxed{x(t) = A(\omega)\cos(\omega t - \delta)}$$

- We now want to plug $x(t)$ into diff. eq. of motion:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t$$

we need this

$$\frac{dx}{dt} = -\omega A(\omega)\sin(\omega t - \delta)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A(\omega)\cos(\omega t - \delta)$$

$$\rightarrow \quad -m\omega^2 A(\omega)\cos(\omega t-\delta) + k A(\omega)\cos(\omega t-\delta) = F_0 \cos\omega t$$

$k = m\omega_0^2$

$F_0 = ka = m\omega_0^2 a$

$$\rightarrow \quad -m\omega^2 A(\omega)\cos(\omega t-\delta) + m\omega_0^2 A(\omega)\cos(\omega t-\delta) = m\omega_0^2 a \cos\omega t$$

↳ expand these terms

↳ divide out

$$\rightarrow \quad -\omega^2 A(\omega)(\cos\omega t\cos\delta + \sin\omega t \sin\delta) + \omega_0^2 A(\omega)(\cos\omega t\cos\delta + \sin\omega t\sin\delta) = \omega_0^2 a \cos\omega t \qquad (*)$$

note textbook error here (shows ω_0 instead)

- How do we simplify this into something we can work with?

→ equate the coefficients of ~~$\cos\delta$~~ $\cos\omega t$ and ~~$\sin\delta$~~ $\sin\omega t$

$$\rightarrow \quad \cos\omega t\left[-\omega^2 A(\omega)\cos\delta + \omega_0^2 A(\omega)\cos\delta\right] = \cos\omega t\left[\omega_0^2 a\right] \qquad (1)$$

$$\& \quad \sin\omega t\left[-\omega^2 A(\omega)\sin\delta + \omega_0^2 A(\omega)\sin\delta\right] = \sin\omega t\,[0] \qquad (2)$$

- As long as our solution satisfies both (1) & (2), it will also satisfy (*) above, which is the eq. of motion we are trying to satisfy.

• Let's work through the math.

(1) $-\omega^2 A(\omega)\cos\delta + \omega_0^2 A(\omega)\cos\delta = \omega_0^2 a$

(divide everything by ω_0^2)

$\rightarrow -\frac{\omega^2}{\omega_0^2} A(\omega)\cos\delta + A(\omega)\cos\delta = a$

$\rightarrow A(\omega)\left[1 - \frac{\omega^2}{\omega_0^2}\right]\cos\delta = a$ [eq. 3.6a in King]

(2) $-\omega^2 A(\omega)\sin\delta + \omega_0^2 A(\omega)\sin\delta = 0$

(divide everything by ω_0^2)

$\rightarrow \frac{-\omega^2}{\omega_0^2} A(\omega)\sin\delta + A(\omega)\sin\delta = 0$

$\rightarrow A(\omega)\left[1 - \frac{\omega^2}{\omega_0^2}\right]\sin\delta = 0$ [eq. 3.6b in King]

• How to proceed from here? If the solution we seek must satisfy both eqs. 3.6a & 3.6b above, then the solution should also satisfy the ratio of $\frac{3.6b}{3.6a}$

$$\rightarrow \frac{A(\omega)\left[1-\frac{\omega^2}{\omega_0^2}\right]\sin\delta}{A(\omega)\left[1-\frac{\omega^2}{\omega_0^2}\right]\cos\delta} = \frac{0}{a} \Rightarrow \frac{\sin\delta}{\cos\delta} = \tan\delta = 0$$

$\rightarrow$ our solution must satisfy this condition

• If $\tan\delta = 0$, the only solutions will have $\delta = 0$ or $\delta = \pi$. These are the only options for the phase difference b/t the driving force and the displacement of the mass (at least in this situation w/ no damping). i.e., the displacement will either be exactly in phase ($\delta=0$) w/ the driving force, or exactly out-of-phase ($\delta=\pi$).

• Let's first consider the case when $\delta=0$. Eq. 3.6a gives:

$$A(\omega)\left[1-\frac{\omega^2}{\omega_0^2}\right]\underbrace{\cos(0)}_{=1}=a$$

$$\rightarrow A(\omega) = \frac{a}{1-\frac{\omega^2}{\omega_0^2}} \quad \text{for } \delta = 0$$

[King eq. 3.7]

• Since we define $A(\omega)$ as a positive quantity, the above only gives $A(\omega) > 0$ (positive #) when $\omega < \omega_0$

→ means that the mass oscillates perfectly in phase ($\delta=0$) with the drive when drive frequency $\omega < \omega_0$, where ω_0 is the natural frequency of oscillation

• Now let's consider what happens when $\delta = \pi$.

Eq. 3.6a gives:

$$A(\omega)\left[1 - \frac{\omega^2}{\omega_0^2}\right]\underbrace{\cos(\pi)}_{=-1} = a$$

$$\rightarrow A(\omega) = \frac{-a}{1 - \omega^2/\omega_0^2} \quad \text{for } \underline{\underline{\delta = \pi}}$$ [King eq. 3.8]

• Here, $A(\omega)$ only positive when $\omega > \omega_0$

→ means that the mass oscillates exactly <u>out-of-phase</u> when the drive frequency $\omega > \omega_0$, where ω_0 is natural frequency of oscillation

• From all the above, we conclude that

$x(t) = A(\omega)\cos(\omega t - \delta)$ is a solution to our eq. of motion $m\frac{d^2x}{dt^2} + kx = F_0\cos\omega t$ and that

$\delta = 0$ for $\omega < \omega_0$

$\delta = \pi$ for $\omega > \omega_0$

- Now that we know $x(t)$ is a solution, let's look at how the amplitude of oscillation $A(\omega)$ varies as a function of drive frequency ω

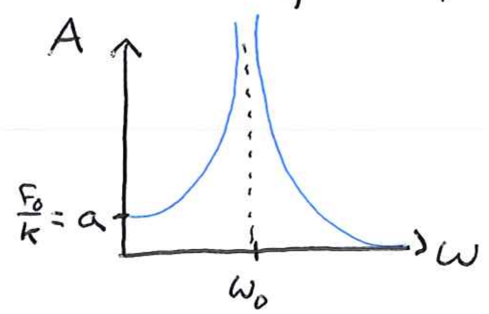
- eq. 3.7 : $A(\omega) = \dfrac{a}{1 - \frac{\omega^2}{\omega_0^2}}$ For $\delta = 0$

  ↳ Here, we see that as $\omega \to 0$, $A(\omega) \to a$ where $a = \dfrac{F_0}{k}$

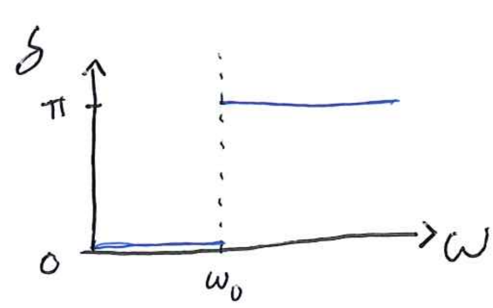
- eq. 3.8 : $A(\omega) = \dfrac{-a}{1 - \frac{\omega^2}{\omega_0^2}}$ for $\delta = \pi$

  ↳ as $\omega \to \infty$, $A(\omega) \to 0$

- From both eqs. 3.7, 3.8, see that as $\omega \to \omega_0$, $A(\omega) \to \infty$



($A(\omega) \to \infty$ is unphysical and is only b/c we included no damping, which will make $A(\omega = \omega_0)$ Finite.)



- Phase difference δ abruptly changes at ω_0

Forced oscillations with damping

- All real harmonic oscillators will have damping, which modifies the eq. of motion:

$$m\frac{d^2x}{dt^2} + \underbrace{b\frac{dx}{dt}} + kx = F_0\cos\omega t$$

↳ recall this is coming from assumption that damping force is proportional to velocity $F_d = -bv$

- making substitutions $\frac{b}{m} = \gamma$, $\frac{k}{m} \equiv \omega_0^2$, we get:

$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t}$$

- Again, assume our ansatz $x(t) = A(\omega)\cos(\omega t - \delta)$ will work

- We also have $\frac{dx}{dt} = -\omega A(\omega)\sin(\omega t - \delta)$

$\omega_0^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} = -\omega^2 A(\omega)\cos(\omega t - \delta)$$

$F_0 = ka = m\omega_0^2 a$

$$\rightarrow -\omega^2 A(\omega)\underline{\cos(\omega t-\delta)} - \gamma\omega A(\omega)\underline{\sin(\omega t - \delta)} + \omega_0^2 A(\omega)\underline{\cos(\omega t-\delta)}$$

$$= \frac{F_0}{m}\cos\omega t$$

Expand these terms

$$\rightarrow -\omega^2 A(\omega)(\cos\omega t\cos\delta + \sin\omega t\sin\delta)$$

$$- \gamma\omega A(\omega)(\sin\omega t\cos\delta - \cos\omega t\sin\delta)$$

same eq. as before except we now have this term

$$+ \omega_0^2 A(\omega)(\cos\omega t\cos\delta + \sin\omega t\sin\delta) = \omega_0^2 a\cos\omega t \quad (**)$$

- How do we simplify this into something we can work with?

$\rightarrow$ Again, equate coefficients of $\cos\omega t$ and $\sin\omega t$

$$\rightarrow \cos\omega t\left[-\omega^2 A(\omega)\cos\delta + \gamma\omega A(\omega)\sin\delta + \omega_0^2 A(\omega)\cos\delta\right] = \cos\omega t\left[\omega_0^2 a\right] \quad (3)$$

$$\rightarrow \sin\omega t\left[-\omega^2 A(\omega)\sin\delta - \gamma\omega A(\omega)\cos\delta + \omega_0^2 A(\omega)\sin\delta\right] = \sin\omega t\left[0\right] \quad (4)$$

- As before, as long as our solution satisfies both (3) & (4), it will also satisfy our eq. (**) above, which is the eq. of motion we are trying to satisfy.

$\longrightarrow$

· Let's work through the math:

(3) $-\omega^2 A(\omega)\cos\delta + \gamma\omega A(\omega)\sin\delta + \omega_0^2 A(\omega)\cos\delta = \omega_0^2 a$

$\rightarrow A(\omega)\left[(\omega_0^2-\omega^2)\cos\delta + \omega\gamma\sin\delta\right] = \omega_0^2 a$ [King eq. 3.11a]

(4) $-\omega^2 A(\omega)\sin\delta - \gamma\omega A(\omega)\cos\delta + \omega_0^2 A(\omega)\sin\delta = 0$

$\rightarrow (\omega_0^2-\omega^2)\sin\delta = \omega\gamma\cos\delta$ [King eq. 3.11b]

$\hookrightarrow \tan\delta = \dfrac{\omega\gamma}{(\omega_0^2-\omega^2)}$ (eq. 3.12)

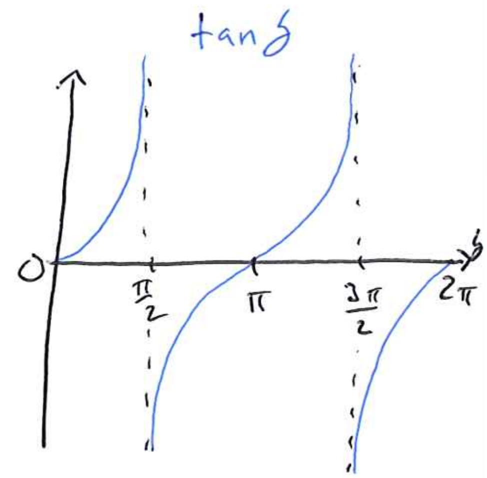
· Our solution <u>must</u> satisfy this condition

· How does this eq. 3.12 behave?

as $\boxed{\omega \rightarrow 0}$, $\tan\delta \rightarrow 0$, $\boxed{\delta \rightarrow 0}$

$\boxed{\omega \rightarrow \infty}$, $\tan\delta \rightarrow \dfrac{1}{-\infty}$, $\boxed{\delta \rightarrow \pi}$

$\boxed{\omega = \omega_0}$, $\tan\delta = \infty$, $\boxed{\delta = \dfrac{\pi}{2}}$



→ Blue and red situations are exactly what we got in the undamped case!

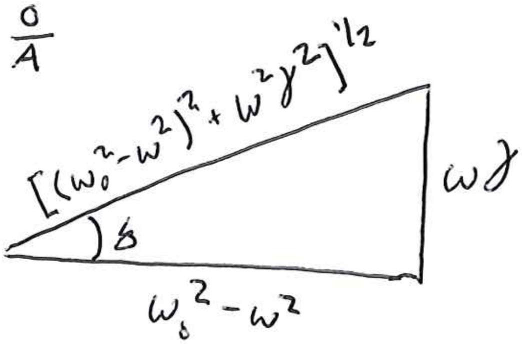
→ On resonance, $\omega = \omega_0$, displacement lags behind drive by $\dfrac{\pi}{2}$!

- This makes sense; w/ $\delta = \frac{\pi}{2}$, mass always moving (ie velocity) in direction of driving force (b/c velocity has $\frac{\pi}{2}$ phase shift from displacement). Similar to pushing a child on a swing.

- Returning to eq. 3.12 $\tan\delta = \dfrac{\omega\gamma}{(\omega_0^2-\omega^2)}$

→ Break this into sin & cos terms using triangle:

$\tan = \frac{O}{A}$



$$\sin\delta = \frac{\omega\gamma}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^{1/2}}$$

$$\cos\delta = \frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2]^2+\omega^2\gamma^2]^{1/2}}$$

- Substitute these into 3.11a, 3.11b, you get (after some simplification)

$$\boxed{A(\omega) = \frac{a\omega_0^2}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^{1/2}} = \frac{F_0/m}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]^{1/2}}}$$

[King eq. 3.15]

→ note that this simplifies to the expression for no damping derived earlier by taking $\gamma \to 0$ ✓

- How does this behave?

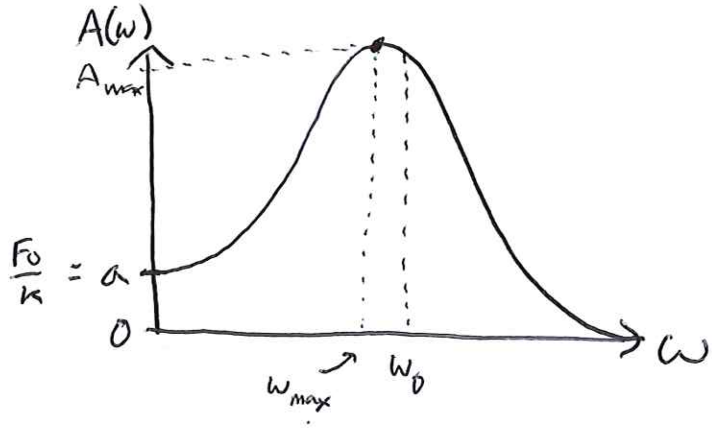
$\omega \to 0, \quad A(\omega) \to a = F_0/k$

$\omega \to \infty, \quad A(\omega) \to 0$

$\omega = \omega_0, \quad A(\omega) = a\omega_0/\gamma$

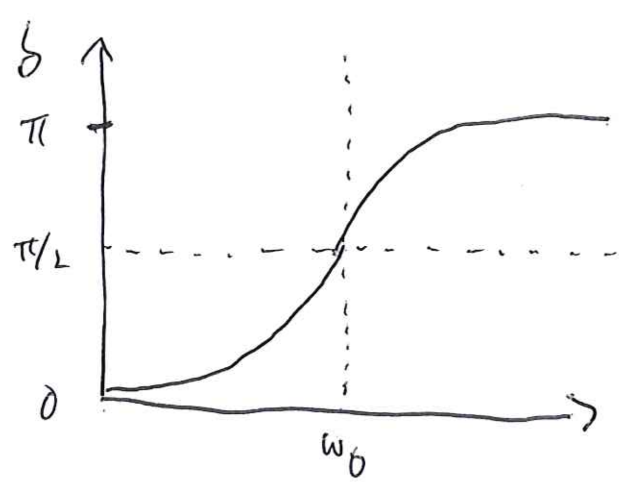
→ Similar results to before, except we see that $A(\omega = \omega_0)$ does not go to infinity but is finite. (But if take $\gamma \to 0$, then <u>does</u> go to ∞)

- Plotting this, we get:



$A(\omega_0) = \frac{a\omega_0}{\gamma}$

But this is not max amplitude!



$\delta(\omega_0) = \pi/2$

• At which ~~point~~ ω is amplitude maximum? How to calculate this?

→ This will be when denominator in $A(\omega)$ expression [King eq. 3.15] is minimal. This occurs when:

$$\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]^{1/2}=0$$ (w/ some math...)

$$\rightarrow \omega = \omega_0\left(1-\gamma^2/2\omega_0^2\right)^{1/2} \equiv \omega_{max}$$

(where $A(\omega)=A_{max}$)

$$\rightarrow \boxed{\omega_{max}^2 = \omega_0^2 - \frac{\gamma^2}{2}}$$

(note difference from DHO w/ $\omega^2=\omega_0^2-\underline{\underline{\frac{\gamma^2}{4}}}$)

$$\rightarrow \boxed{A_{max} = \frac{a\omega_0/\gamma}{\left(1-\gamma^2/4\omega_0^2\right)^{1/2}}}$$

* Note that, similar to DHO, ω_{max} is usually very close to ω_0. So this is a small effect.

⟶

- Amplitude can become extremely large when driven close to $\omega_{max} \simeq \omega_0$, especially when Q-factor $\underbrace{Q \equiv \frac{\omega_0}{\gamma}}$ is large. This is called <u>Resonance</u>

↳ # of complete ~~cyc~~ oscillations before die away

- Using Q-factor, rewrite expressions:

$$\omega_{max} = \omega_0 \left(1 - \frac{1}{2Q^2}\right)^{1/2}$$

$$A_{max} = \frac{aQ}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}}$$

- For light damping $(Q \gg 1)$, $\omega_{max} \simeq \omega_0$, $A_{max} \simeq aQ$

→ Thus, under practical conditions, $\omega_{max} \simeq \omega_0$

→ We also see that a forced HO acts like an amplifier w/ amplification factor Q.